Exercise 3

(a) Find the solution of (1) that is equal to t on the t-axis.

(b) Plot the solution as a function of x and t, and describe the image of the t-axis.

Solution

Equation (1) is

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0. \tag{1}$$

The solution that's equal to t on the t-axis satisfies the following boundary value problem.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \ -\infty < t < \infty$$

$$u(0,t) = t$$

Make the change of variables, $\alpha = x + t$ and $\beta = x - t$, and use the chain rule to write the derivatives in terms of these new variables.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial u}{\partial \alpha}(1) + \frac{\partial u}{\partial \beta}(1) = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = \frac{\partial u}{\partial \alpha}(1) + \frac{\partial u}{\partial \beta}(-1) = \frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}$$

The PDE then becomes

$$\begin{aligned} 0 &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \\ &= \left(\frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}\right) + \left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right) \\ &= 2\frac{\partial u}{\partial \alpha}. \end{aligned}$$

Divide both sides by 2.

$$\frac{\partial u}{\partial \alpha} = 0$$

Integrate both sides partially with respect to α to get u.

$$u(\alpha, \beta) = f(\beta)$$

Here f is an arbitrary function. Now that the general solution to the PDE is known, change back to the original variables.

$$u(x,t) = f(x - t)$$

To determine f, use the boundary condition.

$$u(0,t) = f(-t) = t \quad \Rightarrow \quad f(t) = -t$$

What this actually means is that $f(w) = -w$, where w is any expression, so

$$f(x-t) = -(x-t) = t - x.$$

Therefore,

$$u(x,t) = t - x.$$

Below is a plot of $u(x, t)$ versus x at several moments in time.